PURE SUBGROUPS OF TORSION-FREE GROUPS

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ABSTRACT. In this paper, we show that certain new notions of purity stronger than the classical concept are relevant to the study of torsion-free abelian groups. In particular, implications of ∗-purity, a concept introduced in one of our recent papers, are investigated. We settle an open question (posed by Nongxa) by proving that the union of an ascending countable sequence of ∗-pure subgroups is completely decomposable provided the subgroups are. This result is false for ordinary purity. The principal result of the paper, however, deals with Σ-purity, a concept stronger than ∗-purity but weaker than the usual notion of strong purity. Our main theorem, which has a number of corollaries including the recent result of Nongxa that strongly pure subgroups of separable groups are again separable, states that a Σ-pure subgroup of a k-group is itself a k-group.

Among other results is the negative resolution of the conjecture (valid in the countable case) that a strongly pure subgroup of a completely decomposable group is again completely decomposable.

1. **Introduction.** Using the concept of type, R. Baer [B] classified completely decomposable, torsion-free groups in 1937. (All groups considered are abelian.) The fact that direct summands of completely decomposable groups are again completely decomposable is known as the Baer-Kulikov-Kaplansky Theorem and was proved for the general case only in 1958 by Kaplansky [K]. Consideration of completely decomposable groups leads naturally to what Baer called a separable group. The torsion-free group G is said to be *separable* (in the sense of Baer) if each finite subset of G is contained in a completely decomposable direct summand of G. Pure subgroups of completely decomposable groups are much more general than direct summands. They need not be completely decomposable, and their structure has been determined only in very special cases; see, for example, [R]. In the finite rank case, pure subgroups of completely decomposable groups are generally known as Butler groups and have been studied at length; see, for example, [Bu, A, and AV].

In this paper, we are primarily interested in groups of infinite rank. Our main results are for a concept of purity stronger than the classical concept, but yet general enough to be widely applicable. Indeed, this new concept of purity seems most relevant to the modern theory of torsion-free and mixed groups. For example, the recent results in [N] on strongly pure subgroups become a part of our more general

and, perhaps, more natural theory. Not only do our results reach beyond those in [**N**], but we are able to answer specific questions left open in that paper.

Most of the different versions of purity studied in the literature stem from a homological base, but our new notion of purity is arithmetic in nature.

2. Variances of purity. The classical notion of a subgroup H being pure in the abelian group G is that $nG \cap H = nH$ for each positive integer n. If G is torsion-free, this is equivalent to G/H being again torsion-free. We will consider presently several kinds of purity, but when we say simply that H is pure in G, the above is what we mean. We now consider three stronger versions of purity, and we show how these compare with one another.

Strong purity. A subgroup H of G is strongly pure in G if, for any element $h \in H$, there exists a homomorphism $\phi_h \colon G \to H$ that leaves h fixed. It is well known [**JR**] that if H is strongly pure in G and S is a finite subset of H, then there exists a homomorphism $\phi_S \colon G \to H$ that leaves the elements of S fixed. Thus if H has finite rank and is a strongly pure subgroup of a torsion-free group G, then H is a direct summand of G. This suggests a natural question, which was raised in [**N**] but left open. Must a strongly pure subgroup H of a completely decomposable torsion-free group G again be completely decomposable? It was shown in [**N**] that the answer is in the affirmative for a countable G, but we shall prove here that in general the answer is "no".

Star-purity. The concept of $*$-purity, introduced in [**HM**$_1$], requires some further notation. By a *height sequence s* we understand a sequence $\{s_p\}_{p \in P}$ indexed by the set P of primes where each s_p is either a nonnegative integer or the symbol ∞. The collection of all height sequences forms a lattice when the operations are defined in the obvious pointwise fashion. With each element x of the torsion-free group G we associate its height sequence $|x|$ where $|x|_p$ is the height of x in G at the prime p. Each height sequence s then gives rise to the fully-invariant subgroup $G(s) = \{x \in G \colon |x| \geqslant s\}$. Recall that height sequences s and t are said to be *equivalent*, $s \sim t$, if (adopting the convention $\infty - \infty = 0$) $\sum_{p \in P} |s_p - t_p|$ is finite. A *type* is just an equivalence class of height sequences. We also need the fully-invariant subgroups $G(s*) = \langle x \in G(s) \colon |x| \nsim s \rangle$ and $G(s*, p) = G(s*) + p(G(s))$ where p is a prime. Finally, we say that a pure subgroup H is $*$-pure in G if $H \cap G(s*) = H(s*)$ and $H \cap G(s*, p) = H(s*, p)$ for all height sequences s and all primes p. We remark that if G is completely decomposable and H is a $*$-pure subgroup of finite rank, then it is not difficult to show that H is a direct summand of G (see [**DR**]).

Sigma-purity. This is the new concept of purity that we are introducing in this paper. A subgroup H of an abelian group G is said to be Σ-*pure* provided that $h = g_1 + g_2 + \cdots + g_n$ with $h \in H$ and $g_i \in G(s_i)$ implies that $h = h_1 + h_2 + \cdots + h_n$ where $h_i \in H(s_i)$. It is interesting to observe that Σ-purity agrees with ordinary purity in the local case, where only one prime is involved. Moreover, for many purposes, the concept of Σ-purity seems to be the right notion of purity for torsion-free and mixed groups. Indeed, the fact that Σ-purity is most relevant to nonlocal groups (where heights are not linearly ordered) is manifested in §3, where we show that Σ-pure subgroups inherit the property of being a k-group or a

separable group. The latter result—a Σ-pure subgroup of a separable group is again separable—implies the main result of [N]: a strongly pure subgroup of a separable group is again separable. Obviously, all of these results are refinements of the well-known theorem by L. Fuchs [F] that asserts that a summand of a separable group is again separable.

PROPOSITION 2.1. *Strongly pure* $\Rightarrow$ Σ-*pure* $\Rightarrow$ *-*pure* $\Rightarrow$ *pure*.

PROOF. Let H be strongly pure in G and suppose that $h = g_1 + g_2 + \cdots + g_n$, where $h \in H$ and $g_i \in G(s_i)$. Since H is strongly pure, there exists $\phi_h: G \to H$ that leaves h fixed. Hence if $\phi_h(g_i) = h_i$, we have that $h = h_1' + h_2 + \cdots + h_n$, where $h_i \in H(s_i)$ since homomorphisms do not decrease heights.

Now assume that H is Σ-pure but relinquish the hypothesis that H is strongly pure. If $h \in H \cap G(s)$ then, by definition of Σ-purity, $h \in H(s)$ and at least H is pure in G. Suppose now that $h \in G(s^*)$. Then $h = g_1 + g_2 + \cdots + g_n$ where $|g_i| = s_i \geqslant s$ and $s_i \nsim s$. Applying the Σ-purity of H, we obtain $h = h_1 + h_2 + \cdots + h_n$ where $|h_i| \geqslant s_i$. Therefore, $h \in H(s^*)$. Similarly $H \cap G(s^*, p) = H(s^*, p)$, so H is *-pure in G. Finally, in order to be *-pure, H is explicitly required to be pure. This completes the proof of the proposition.

REMARK. The results and examples contained herein will demonstrate that none of the implications of the preceding proposition is reversible.

3. Pure subgroups of completely decomposable groups.

As promised above, we shall resolve an open question from [N] by showing that a strongly pure subgroup of a completely decomposable group, though necessarily a separable group, need not itself be completely decomposable. In order to establish this, it is convenient (perhaps essential) to have in our possession a new criterion that a pure subgroup must satisfy if it is completely decomposable. Precisely, we need a special case of a result that appears in [AH] and, for sake of completeness and for the convenience of the reader, we give a simplified proof here of the version we require.

THEOREM 3.1 [AH]. *Let H be a pure subgroup of a torsion-free group G. In order for H to be completely decomposable it must be a separable subgroup (in the sense of Hill).*

PROOF. First we need to define exactly what a separable subgroup means (in the sense of Hill [H_1, AH]). A subgroup H of G is *separable* in G if for each $g \in G$ there is a countable subset $\{h_n\}_{n < \omega}$ of H satisfying the following condition: for $h \in H$, there is a corresponding n such that $|g + h| \leqslant |g + h_n|$.

Suppose that H is completely decomposable and appears in the torsion-free group G as a pure subgroup. Assume that H is not separable in G. Since H is not separable, there is a $g \in G$ such that for each countable subset C of H there is a corresponding element $h_C \in H$ with the property that $|g + x| \geqslant |g + h_C|$ fails for each x in C. Letting $H = \oplus_{i \in I} A_i$, where A_i has rank 1, we can choose an ascending sequence of countable subsets $J(n)$ of I, with $J(0) = \varnothing$, such that the following condition is satisfied where $H_n = \oplus_{j \in J(n)} A_j$: If $h_n \in H_n$ and the inequality $|g + h_n|_p \nleqq |g + h|_p$ holds for some $h \in H$ and prime p, then the inequality

already holds for some $h \in H_{n+1}$. [Here, "$\underset{\neq}{\leqslant}$" is appropriate rather than "$<$" due to the convention that $\infty < \infty$.]

Set $H_\omega = \bigcup_{n<\omega} H_n = \bigoplus_{j \in J(\omega)} A_j$ where $J(\omega) = \bigcup_{n<\omega} J(n)$. Since H_ω is countable, there exists $h \in H$ such that $|g + x| \geqslant |g + h|$ fails for every x in H_ω. In particular, $|g + h'| \geqslant |g + h|$ fails where h' is the projection of h onto H_ω and consequently $|g + h'|_p \underset{\neq}{\leqslant} |g + h|_p$ for some prime p. Thus, according to the preceding condition, there is already some h'' in H_ω such that $|g + h'|_p \underset{\neq}{\leqslant} |g + h''|_p$. It then follows that

$$|h' - h''|_p = |g + h'|_p \underset{\neq}{\leqslant} |g + h|_p \wedge |g + h''|_p \leqslant |h - h''|_p.$$

But this is absurd since the choice of h' as the projection of h onto H_ω implies that $|h' - x|_p \geqslant |h - x|_p$ for all $x \in H_\omega$. We conclude, therefore, that the assumption that H is not separable in G must be false, and the theorem follows.

We are now prepared to answer Nongxa's question.

THEOREM 3.2. *Strongly pure subgroups of completely decomposable groups need not be completely decomposable.*

PROOF. Denote by S the set of all height sequences with no ∞'s. For each $s \in S$, let G_s be a rank 1 torsion-free group with an element g_s whose height sequence in G_s is exactly s. Set $G = \bigoplus_{s \in S} G_s$. Denote by ϕ_s the map from G_s to the group of rationals Q that takes g_s onto 1, and let $\phi = \Sigma \phi_s$. Observe that $\phi \colon G \to Q$ is epic (because all height sequences s are used except those with ∞'s). Let H be the kernel of ϕ and suppose $h \in H$. Since $h \in G$, we can write, for some positive integer m,

$$mh = m_1 g_{s(1)} + m_2 g_{s(2)} + \cdots + m_r g_{s(r)}$$

where $s(1), s(2), \ldots, s(r)$ are distinct elements of S and $m_i \in Z$. Observe that

$$\sum_{i \leqslant r} m_i = \sum_{i \leqslant r} m_i \phi(g_{s(i)}) = \phi(mh) = 0.$$

Now select $t \in S$ so that $t \geqslant s(i)$ and $t \nrightarrow s(i)$ for each $i \leqslant r$. Since $\Sigma_{i<r} m_i = 0$, we have $mh = \Sigma_{i \leqslant r} m_i(g_{s(i)} - g_t)$ where each $g_{s(i)} - g_t$ is in H. Moreover, by Corollary 2.6 in [$\mathbf{HM_1}$], we have a direct decomposition

$$G = \left(\bigoplus_{i \leqslant r} \langle g_{s(i)} - g_t \rangle_* \right) \oplus \left(\bigoplus_{s \neq s(i)} G_s \right)$$

(where, as usual, A_* denotes the pure subgroup generated by A). Since $H_0 = \bigoplus_{i \leqslant r} \langle g_{s(i)} - g_t \rangle_*$ is a finite rank summand of G contained in H and since $h \in H_0$, it quickly follows that H is strongly pure in G; indeed, ϕ_h is simply the projection of G onto H_0. In order to show that H is not completely decomposable, we employ Theorem 3.1. In other words, we show that H is not separable in G.

Assume that H is separable in G. Let s denote the height sequence consisting entirely of zeroes and let $g = g_s$. According to the assumption that H is separable in G, there exists a countable number of elements $h_1, h_2, \ldots, h_n, \ldots$ in H such that if $h \in H$ there exists n (depending on the choice of h) with the property that $|g + h| \leqslant |g + h_n|$. Observe that no nonzero element of G can have an infinity in its height sequence. Since both g and the h_n's are fixed, the height sequences $|g + h_n|$

are fixed, for each n, and have no ∞'s. Select a height sequence $t' \in S$ so that, at the nth prime, t' exceeds $|g + h_n|$; for example, if p is the nth prime,

$$t'_p = |g + h_n|_p + 1.$$

Finally, let $h' = g_{t'} - g = g_{t'} - g_s$. Observe that $|g + h'| = |g_{t'}|$ and that this contradicts the supposition that $|g + h'| \leq |g + h_n|$ for some n. Therefore, H is not separable in G and according to Theorem 3.1 cannot be completely decomposable.

4. Σ-pure subgroups of k-groups. Torsion-free k-groups, introduced in [**HM$_1$**], can be defined as those torsion-free groups in which each finite subset is contained in a completely decomposable $*$-pure subgroup (see [**DR**]). But we prefer to return to the original definition in terms of primitive elements and $*$-valued coproducts. A nonzero element x of the torsion-free group G is said to be *primitive* if $x \notin G(s^*, p)$ for all $s \sim |x|$ with $s_p = |x|_p \neq \infty$. If G is completely decomposable, then x is primitive if and only if $\langle x \rangle_*$ is a direct summand (see 2.9 in [**HM$_1$**]). A direct sum $A = \oplus_{i \in I} A_i$ of independent subgroups of G is said to be a $*$-*valued coproduct* if $A \cap F = \oplus_{i \in I} A_i \cap F$ for all fully-invariant subgroups F of the form $G(s)$, $G(s^*)$, or $G(s^*, p)$. A pure subgroup H is *knice* in G if for each finite subset S of G there correspond primitive elements $x_1, \ldots, x_n$ such that $H \oplus \langle x_1 \rangle_* \oplus \cdots \oplus \langle x_n \rangle_*$ is a $*$-valued coproduct containing S. Once the preceding concepts have emerged the notion of a k-group is imperative, for G is a *k-group* if 0 is a knice subgroup. A fundamental fact about k-groups that we shall use repeatedly is the following: If $H = \langle x_1 \rangle_* \oplus \cdots \oplus \langle x_n \rangle_*$ is a $*$-valued coproduct where each x_i is a primitive element of the k-group G, then H is a pure knice subgroup of G (see Proposition 4.2 in [**HM$_1$**]).

If H is a pure knice subgroup of G, then H is Σ-pure in G. Indeed suppose $h = g_1 + \cdots + g_n$ is in H and let $S = \{g_1, \ldots, g_n\}$. Then, by definition of kniceness, there is a $*$-valued coproduct $H \oplus A$ in G that contains S. Writing $g_i = h_i + a_i$ where $h_i \in H$ and $a_i \in A$, we see that $|g_i| \leq |h_i|$ and $h = h_1 + \cdots + h_n$. One should note, however, that kniceness is not an inductive property.

In this section, we generalize the results of [**N**] in two directions. First, our results apply to k-groups, not just separable groups. Second, our results are for Σ-pure subgroups, not just strongly pure subgroups. The following is a simple example that demonstrates that the generalizations are genuine in both directions.

Let $G = \langle \oplus_{\aleph_0} Z, 2\Pi_{\aleph_0} Z \rangle$. Since G is $\aleph_1$-free, all nonzero elements are primitive and if the direct sum $F = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is pure in G, then it is a $*$-valued coproduct. Thus G is a k-group, but, as is well known, not a separable group [**G**, p. 114]. In fact the element $x = (2, 2, \ldots, 2, \ldots)$ is contained in no finite rank direct summand. It follows that $H = \langle x \rangle = \langle x \rangle_*$ is knice in G and hence Σ-pure, but H is not strongly pure since it fails to be a direct summand.

We are now ready for our main theorem.

THEOREM 4.1. *A Σ-pure subgroup of a torsion-free k-group is again a k-group.*

PROOF. Let H be a Σ-pure subgroup of the torsion-free k-group G. Since, in particular, H is $*$-pure in G, elements in H are primitive if and only if they are primitive in G and a direct sum of subgroups in H is a $*$-valued coproduct in H if

and only if it is a $*$-valued coproduct in G. If $A = \langle a_1 \rangle_* \oplus \cdots \oplus \langle a_m \rangle_*$ is a $*$-valued coproduct with the a_i's primitive in G, then we shall refer to A as a *finite rank free $*$-valued subgroup*. Now let S be a finite subset of H. To show that H itself is a k-group, it suffices to construct a finite rank free $*$-valued subgroup A of G such that $S \subseteq A \subseteq H$. Proceeding by induction on $|S|$, we may assume that all but one element of S, say h, can be imbedded in such an A. But since A is necessarily a pure knice subgroup of G, we know that h is contained in a $*$-valued coproduct $A \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n$ with the following conditions satisfied for each $i \leqslant n$.

   (a) G_i is homogeneous of type τ_i.

   (b) G_i is a finite rank free $*$-valued subgroup of G.

Moreover, since $A \subseteq H$, there is no loss of generality in assuming

   (c) $h \in G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

Let us also label our original condition as (d).

   (d) $A \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n$ is a $*$-valued coproduct in G.

Our proof will be completed by showing that with a judicious choice of the G_i's, there will be a finite rank free $*$-valued subgroup B contained in $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ such that $h \in B \subseteq H$; for under these circumstances S will be contained in the finite rank free $*$-valued subgroup $A \oplus B \subseteq H$.

There is no reason, of course, to expect that our original choice of the G_i's will have this desired property. Thus we must allow ourselves the freedom to enlarge, at most finitely many times, the G_i's until the construction of the desired B is possible. We shall, however, always adhere to the restrictions (a)–(d) and so, in particular, n and the types $\tau_1, \tau_2, \ldots, \tau_n$ remain fixed throughout our proof. Furthermore, whenever we replace G_i by a larger G_i', we insist that $G_i' = G_i \oplus C_i$ be a $*$-valued coproduct with C_i a finite rank homogeneous free $*$-valued subgroup of type τ_i. The key to our being able to make such enlargements is the fact that, at any stage, the subgroup $A \oplus \dot{G}_1 \oplus G_2 \oplus \cdots \oplus G_n$ will be a pure knice subgroup of G. As the types τ_i shall not vary, we agree once and for all to arrange them in such an order that τ_i is minimal in the set $\{ \tau_i, \tau_{i+1}, \ldots, \tau_n \}$.

The ultimate construction of $B \subseteq G_1 \oplus G_2 \oplus \cdots \oplus G_n$ depends on a secondary induction. So let $0 \leqslant r < n$ and suppose we have found elements $h_1, h_2, \ldots, h_r$ in H, together with an auxiliary subgroup C, such that, in conjunction with a suitable choice of the G_i's satisfying (a)–(d), we have the following five conditions satisfied.

   (1) C is a finite rank free $*$-valued subgroup each homogeneous component of which is of type distinct from any of the τ_i's.

   (2) $A \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$ is a $*$-valued coproduct in G.

   (3) For each $i \leqslant r$, $h_i \in G_i \oplus G_{i+1} \oplus \cdots \oplus G_n \oplus C$.

   (4) For each $i \leqslant r$, $|h_i| = |g_i|$ where g_i is the projection of h_i in G_i.

   (5) $h - \sum_{i \leqslant r} h_i = y_{r+1} + y_{r+2} + \cdots + y_n$ where $y_j \in H$ and $\text{type}(y_j) \geqslant \tau_j$ for $j = r + 1, r + 2, \ldots, n$.

If $r = 0$, it is understood that $C = 0$ and that conditions (3) and (4) are vacuous. Moreover, for $r = 0$, (2) and (5) reduce to (d) and (c), respectively. Hence the induction hypotheses hold for $r = 0$.

Assuming $r < n$, we now show that these hypotheses also hold for $r + 1$. By (5), we can write $h - \sum_{i \leqslant r} h_i = y_{r+1} + y_{r+2} + \cdots + y_n$ where type$(y_j) \geqslant \tau_j$. Since $A \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$ is knice in G, by an appropriate enlargement of C and the G_i's, we may suppose without loss of generality that $y_{r+1}, y_{r+2}, \ldots, y_n$ are actually contained in this subgroup. Since $h - \sum_{i \leqslant r} h_i$ is in $G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$ and since the conditions type$(y_j) \geqslant \tau_j$ preclude any of the elements $y_{r+1}, \ldots, y_n$ from having nonzero projections in $G_1, \ldots, G_r$, we may write

(i) $h - \sum_{i \leqslant r} h_i = g_{r+1} + g_{r+2} + \cdots + g_n + c$,

where $g_j \in G_j$ for $j = r + 1, r + 2, \ldots, n$ and $c \in C$. Furthermore, considering the projections into C of the individual y_j's, we see that $c = d_{r+1} + d_{r+2} + \cdots + d_n$ where type$(d_j) > \tau_j$ for each j. Next, utilizing the fact that H is Σ-pure in G, we write

(ii) $h - \sum_{i \leqslant r} h_i = x_{r+1} + x_{r+2} + \cdots + x_n + w_{r+1} + w_{r+2} + \cdots + w_n$ where the x_j's and w_j's are in H with $|g_j| \leqslant |x_j|$ and $|d_j| \leqslant |w_j|$ for $j = r + 1, r + 2, \ldots, n$. Once again, enlarging C and the G_i's, we may assume that the x_j's and w_j's are also contained in $A \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$. In fact, since $A \subseteq H$, all the above conditions are satisfied if we replace the x_j's and w_j's by their projections in $G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$. Moreover, because of the arrangement of the τ_i's and the conditions on the types of the x_j's and w_j's, these elements must in fact lie in $G_{r+1} \oplus G_{r+2} \oplus \cdots \oplus G_n \oplus C$ and, among them all, only x_{r+1} can have nonzero projection into G_{r+1}. It follows, comparing equations (i) and (ii), that g_{r+1} is the projection of x_{r+1} into G_{r+1} and, consequently, that $|x_{r+1}| = |g_{r+1}|$.

Now set $h_{r+1} = x_{r+1}$ and observe that the only thing keeping $h_1, h_2, \ldots, h_r, h_{r+1}$ from satisfying the five conditions of the secondary induction hypotheses is the unfortunate appearance of the term w_{r+1} in equation (ii). Nevertheless, it is clear that we can write

$$h - \sum_{i \leqslant r+1} h_i = g'_{r+2} + \cdots + g'_n + c_1 + c_2 + \cdots + c_{r+1}$$

where $g'_j \in G_j$ for $j = r + 2, \ldots, n$ and c_i is the projection of $-h_i$ into C for $i = 1, 2, \ldots, r + 1$. Therefore $|h_i| \leqslant |c_i|$ and, because of conditions (1), $c_i \in G(s_i *) \subseteq G(\tau_i *)$ where $s_i = |h_i|$. Once again we apply the fact that H is Σ-pure in G and write

$$h - \sum_{i \leqslant r+1} h_i = y'_{r+2} + \cdots + y'_n + z_1 + z_2 + \cdots + z_{r+1}$$

where the y'_j's and z_i's are in H with $|g'_j| \leqslant |y'_j|$ for $j = r + 2, \ldots, n$, and $z_i \in G(s_i *) \subseteq G(\tau_i *)$ for $i = 1, 2, \ldots, r + 1$. As before, enlarging C and the G_i's, we may assume that the y'_j's and z_i's are in $G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$. Notice furthermore that $z_i \in G(\tau_i *)$ implies that $z_i \in G_{i+1} \oplus G_{i+2} \oplus \cdots \oplus G_n \oplus C$. Finally, we let $h'_i = h_i - z_i$ for $i = 1, 2, \ldots, r + 1$ and observe that the projection of h'_i in G_i is still g_i. Moreover, by 2.3, 2.5 and 2.7 of [$\mathbf{HM_1}$], $|h'_i| = |h_i| = |g_i|$. Hence $h'_1, h'_2, \ldots, h'_{r+1}$ satisfy all five conditions of the secondary induction.

Having completed the secondary induction, we conclude that $h = h_1 + h_2 + \cdots + h_n$ with $h_i \in H \cap (G_i \oplus G_{i+1} \oplus \cdots \oplus G_n \oplus C)$ and $|h_i| = |g_i|$ where g_i is the projection of h_i in G_i. Then, by 2.5 and 2.7 of [$\mathbf{HM_1}$], the h_i's are primitive

elements (or else 0) and a succession of applications of 2.5 [**HM**$_1$] lead to the desired conclusion that $B = \langle h_1 \rangle_* \oplus \langle h_2 \rangle_* \oplus \cdots \oplus \langle h_n \rangle_*$ is a $*$-valued coproduct. What is more, $A \oplus \langle h_1 \rangle_* \oplus \langle h_2 \rangle_* \oplus \cdots \oplus \langle h_n \rangle_*$ is a $*$-valued coproduct in H containing S, and the proof of Theorem 4.1 is complete.

COROLLARY 4.2. *A strongly pure subgroup of a k-group is again a k-group.*

COROLLARY 4.3 [**N**]. *A strongly pure subgroup of a separable group is again a separable group.*

PROOF. By Proposition 2.1 and Theorem 4.1, a strongly pure subgroup H of a separable group G is at least a k-group. To show that H is separable it suffices by Theorem 4.3 of [**HM**$_1$] to argue that any pure and knice subgroup A of H having finite rank is a direct summand of H. But, in fact, A is a direct summand of G.

COROLLARY 4.4. *A Σ-pure subgroup of a separable group is strongly pure.*

COROLLARY 4.5 [**HM**$_1$, **F**]. *A direct summand of a k-group or a direct summand of a separable group is again, respectively, a k-group or separable.*

COROLLARY 4.6 [**HM**$_1$, **DR**]. *A pure knice subgroup of a k-group is again a k-group.*

We close this section with one further general observation concerning Σ-pure subgroups of k-groups.

PROPOSITION 4.7. *If G is a k-group, then every infinite subset can be imbedded in a Σ-pure subgroup of the same cardinality.*

PROOF. Since Σ-purity is an inductive property, it suffices to show that an arbitrary countable subset S of the k-group G can be imbedded in a countable Σ-pure subgroup. Let $S = \bigcup_{n < \omega} S_n$ where each S_n is finite. We need only prove that there exists an ascending sequence $\{H_n\}_{n < \omega}$ of finite rank pure knice subgroups of G such that $S_n \subseteq H_n$ for each $n < \omega$; for then $H = \bigcup_{n < \omega} H_n$ will be a countable Σ-pure subgroup containing S. The existence of a finite rank pure knice subgroup $H_0 \supseteq S_0$ follows immediately from Proposition 4.2 in [**HM**$_1$] and the fact that G is a k-group. Then given $H_0, \ldots, H_n$ with the desired properties, the kniceness of H_n, implies that there are primitive elements $x_1, \ldots, x_m$ such that $H_{n+1} = H_n \oplus \langle x_1 \rangle_* \oplus \cdots \oplus \langle x_m \rangle_*$ is a $*$-valued coproduct containing S_{n+1}. But again using 4.2 of [**HM**$_1$], we conclude that H_{n+1} is also a pure knice subgroup of G. Thus, the desired sequence $\{H_n\}_{n < \omega}$ exists by induction, and the proof is complete.

COROLLARY 4.8 [**DR**]. *An uncountable k-group is the smooth union of k-groups of smaller cardinality.*

5. Unions of $*$-pure subgroups. Another question raised in [**N**] but left open is whether or not a torsion-free group which is the ascending union of a sequence of strongly pure completely decomposable subgroups is itself completely decomposable. An affirmative answer is given below under the weaker assumption that the subgroups are $*$-pure. Recall that G is said to be an *fσ-union* of the family of

subsets $\{G_\alpha\}_{\alpha \in A}$ if for each finite subset S of G there exists an α such that $S \subseteq G_\alpha$. In particular, an ascending union is an $f\sigma$-union. This section is devoted to the proof and some consequences of the following theorem.

THEOREM 5.1. *If the torsion free group G is the $f\sigma$-union of a sequence $\{G_n\}_{n < \omega}$ of $*$-pure completely decomposable subgroups, then G is completely decomposable.*

To place this result in the same context as the analogous theorems established in $[\mathbf{H_2}]$ for totally projective groups and in $[\mathbf{HM_2}]$ for p-local Warfield groups, we shall base its proof on the Axiom 3 characterization of completely decomposable groups obtained in $[\mathbf{HM_1}]$.

DEFINITION 5.2. A family $\mathscr{C}$ of subgroups of G is said to be an Axiom 3 family if the following three conditions are satisfied:

(i) $0 \in \mathscr{C}$.

(ii) If $A_i \in \mathscr{C}$ for each $i \in I$, then $\sum_{i \in I} A_i \in \mathscr{C}$.

(iii) If $A \in \mathscr{C}$ and S is a countable subset of G, then there is a $B \in \mathscr{C}$ such that $\langle S, A \rangle \subseteq B$ and B/A is countable.

In Theorem 5.3 of $[\mathbf{HM_1}]$, we show that a torsion-free group G is completely decomposable if and only if there is an Axiom 3 family $\mathscr{C}$ consisting of pure knice subgroups of G. Indeed, as the proof of that theorem demonstrates, it suffices to exhibit a family $\mathscr{C}$ of pure knice subgroups satisfying (i), (iii) and the weaker condition

(ii′) $\mathscr{C}$ is closed under the union of chains.

In broad outline, we shall prove Theorem 5.1 by selecting, for each $n < \omega$, an Axiom 3 family $\mathscr{C}_n$ of pure knice subgroups of G_n and then utilize all these $\mathscr{C}_n$'s to extract from G a family $\mathscr{C}$ of pure knice subgroups satisfying (i), (ii′) and (iii). In fact, the description of $\mathscr{C}$ is quite simple. We take $\mathscr{C}$ to consist of all subgroups A of G which satisfy the following two conditions:

(1) A is a pure subgroup of G such that $A \cap G_n \in \mathscr{C}_n$ for all $n < \omega$.

(2) For each $n < \omega$, $(A + G_n) \cap F = (A \cap F) + (G_n \cap F)$ for all fully invariant subgroups F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$ where s is any height sequence and p is an arbitrary prime.

Notice that (2) is clearly an inductive property and, in view of (ii), so is (1). Thus it is evident that $\mathscr{C}$ satisfies (i) and (ii′).

It is also easy to prove that each $A \in \mathscr{C}$ is a knice subgroup of G. Indeed let S be a finite subset of G and choose $n < \omega$ such that $S \subseteq G_n$. By (1), there exist primitive elements $x_1, \ldots, x_m$ in G_n and a $*$-valued coproduct $N = (A \cap G_n) \oplus \langle x_1 \rangle \oplus \cdots \oplus \langle x_m \rangle$ which contains S. Because of the $*$-purity of G_n, this is actually a $*$-valued coproduct in G and, moreover, the x_i's are primitive in G. Taking $C = \langle x_1, \ldots, x_m \rangle$, it suffices to argue that $N' = A \oplus C$ is a $*$-valued coproduct in G, that is, we need to show that $(A \oplus C) \cap F = (A \cap F) \oplus (C \cap F)$ for all F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$. Let us suppose then that $a + c \in F$ where $a \in A$, $c \in C$ and F is any such fully invariant subgroup. By (2), $a + c = a_1 + x$ where $a_1 \in A \cap F$ and $x \in G_n \cap F$. Then $a - a_1 = x - c$ is an element of $A \cap G_n$ such that $(a - a_1) + c = x$ is in $((A \cap G_n) \oplus C) \cap F$. But since $N = (A \cap G_n) \oplus C$

is a $*$-valued coproduct in the $*$-pure subgroup G_n and since $a_1 \in F$, it follows that $a \in A \cap F$ and $c \in C \cap F$, as desired.

All that remains to complete the proof of Theorem 5.1 is to verify that $\mathscr{C}$ satisfies the countable extension property (iii). Toward this end, we require a lemma which quite likely will prove useful in future applications. In keeping with the terminology and notation of [$\mathbf{H_1}$], we say that the subgroups A and H are *compatible* and write $A\|H$ provided for each pair $(a, h) \in A \times H$ there exists an $h' \in A \cap H$ such that $|a + h'| \geqslant |a + h|$. As our usage suggests, compatibility is a symmetric relation and, in fact, it is readily checked that $A\|H$ for subgroups of the torsion-free group G if and only if $(A + H) \cap G(s) = (A \cap G(s)) + (H \cap G(s))$ for all height sequences s.

LEMMA 5.3. *If S is a countable subset of the torsion-free group G and if the subgroup H is separable (in the sense of Hill), then there is a countable pure subgroup A of G such that $A \supseteq S$ and $(A + H) \cap F = (A \cap F) + (H \cap F)$ for all fully invariant subgroups F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$.*

PROOF. First we show that we can at least find a countable pure subgroup $A \supseteq S$ such that $A\|H$. Actually it suffices to construct an ascending sequence $\{A_n\}_{n < \omega}$ of countable pure subgroups such that $A_0 = \langle S \rangle_*$ and, for each $n < \omega$, if $(a, h) \in A_n \times H$, then there is an $h' \in A_{n+1} \cap H$ such that $|a + h'| \geqslant |a + h|$; for then $A = \bigcup_{n < \omega} A_n$ will be compatible with H. Suppose then that $A_0, \ldots, A_n$ having the desired properties have been constructed and let $a_1, a_2, \ldots, a_m, \ldots$ be an enumeration of the elements of A_n. Since H is separable in G, we can associated with each $a_m \in A_n$ a corresponding sequence $\{h_{m,i}\}_{i < \omega}$ in H with the following property: If $h \in H$, then $|a_m + h| \leqslant |a_m + h_{m,i}|$ for some $i < \omega$. Now take A_{n+1} to be the pure closure of the subgroup of G generated by A_n and the doubly infinite collection of all the $h_{m,i}$'s. It is clear that A_{n+1} satisfies the requisite condition and hence the construction of the desired sequence $\{A_n\}_{n < \omega}$ follows by induction.

Next we show that given a countable subgroup A of G with $A\|H$, we can find a countable subgroup $B \supseteq A$ such that $(A + H) \cap F \subseteq (B \cap F) + H$ for all fully invariant subgroups F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$. For notational convenience, let $G(s^*, 0) = G(s^*)$ and take $\mathscr{F}$ to consist of all fully invariant subgroups F of the above forms for which $(A + H) \cap F \not\subseteq (A \cap F) + H$. Since $A\|H$, each $F \in \mathscr{F}$ is of the form $G(s^*, p)$ for some height sequence s and some p in $\mathbf{P} \cup \{0\}$. Now for each $F = G(s^*, p)$ in $\mathscr{F}$, let A_F consist of all elements $a \in A \cap G(s)$ such that $|a| \sim s$ and $(a + H) \cap F \not\subseteq (A \cap F) + H$. We shall first show that $A_F \neq \varnothing$ for each $F \in \mathscr{F}$, which (because of the countability of A, $\mathbf{P}$ and any given type σ) will imply that $\mathscr{F}$ is countable. Indeed suppose $F = G(s^*, p)$ is in $\mathscr{F}$ and we have $a \in A$ and $h \in H$ such that $a + h$ is in F but not in $(A \cap F) + H$. Since $A\|H$, we may assume that $a \in A \cap G(s)$. But then it follows that $|a| \sim s$, since otherwise we would have $a + h \in (A \cap G(s^*)) + H \subseteq (A \cap F) + H$. Thus $A_F \neq \varnothing$, as claimed. Now for each $F \in \mathscr{F}$ and each $a \in A_F$, fix some $h_a \in H$ such that $a + h_a \in F$ and $h_a \notin F$. Since $\mathscr{F}$ and each A_F is countable, the definition of the $G(s^*, p)$'s implies that there is a countable subgroup $B \supseteq A$ such

that $a + h_a \in B \cap F$ for all $F \in \mathscr{F}$ and all $a \in A_F$. It is then trivial that $(A + H) \cap F \subseteq (B \cap F) + H$ for all fully invariant F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$.

Finally, to obtain a countable subgroup A satisfying our lemma, we apply the constructions of the last two paragraphs to build up a nested sequence $\langle S \rangle \subseteq A_1 \subseteq B_1 \subseteq \cdots \subseteq A_n \subseteq B_n \subseteq \cdots$ of countable subgroups such that, for each n, A_n is pure in G, $A_n \| H$ and $(A_n + H) \cap F \subseteq (B_n \cap F) + H$ for all fully invariant subgroups F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$. Then the subgroup $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ obviously has the desired property, and the proof of the lemma is complete.

We are now in position to finish the proof of Theorem 5.1 by showing that our $\mathscr{C}$ satisfies (iii) of Definition 5.2. So let A be a fixed element of $\mathscr{C}$ and observe that each quotient group $G_n / A \cap G_n$ is completely decomposable since

$$\mathscr{C}_n' = \{ C / A \cap G_n : C \in \mathscr{C}_n \text{ and } C \supseteq A \cap G_n \}$$

is an Axiom 3 family of pure knice subgroups. Also since A is pure in G and $A \| G_n$, it follows that $A + G_n$ is pure in G. Thus, by Theorem 3.1, each of the completely decomposable groups $A + G_n / A$ is separable (in the sense of Hill) in the group G/A. For the moment, let us fix $n < \omega$ and take $H = G_n$. Since A is a pure knice subgroup of G, it is balanced in G by Theorem 4.3 in [$\mathbf{HM}_1$]. Consequently, $(G/A)(s) = G(s) + A/A$ for all height sequences s and, of course, similar relations hold for $(G/A)(s^*)$ and $(G/A)(s^*, p)$. Because $A + H/A = A + G_n/A$ is separable in G/A, Lemma 5.3 allows us to select, for any given countable subset S of G, a countable subgroup B_1/A of G/A with $B_1 \supseteq \langle S, A \rangle$ and such that

$$(B_1/A + (A + H)/A) \cap (F + A/A) = ((B_1 \cap F) + A/A) + ((H \cap F) + A/A)$$

for all fully invariant subgroups F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$. But since A satisfies (2) and $H = G_n$, it follows readily that $(B_1 + H) \cap F = (B_1 \cap F) + (H \cap F)$ for all such F's. Then, for our fixed n, choose $C_1 \in \mathscr{C}_n$ such that $C_1 \supseteq B_1 \cap G_n$ and $C_1/A \cap G_n$ is countable. (This is possible because $\mathscr{C}_n$ satisfies (iii) and B_1/A is countable.) Continuing in this manner, we construct two ascending sequences of subgroups $\{ B_m \}_{m=1}^{\infty}$ and $\{ C_m \}_{m=1}^{\infty}$ such that, for all m, B_m/A and $C_m/A \cap G_n$ are countable, $B_m \cap G_n \subseteq C_m \subseteq B_{m+1}$, $C_m \in \mathscr{C}_n$ and $(B_m + H) \cap F = (B_m \cap F) + (H \cap F)$ for all fully-invariant subgroups F of the form $G(s)$, $G(s^*)$ or $G(s^*, p)$. If we now take B to be the union of the B_m's and C the union of the C_m's, we will have $B \cap G_n = C$ and B/A countable. Moreover, since $\mathscr{C}_n$ satisfies (ii), the subgroup B will, for our fixed choice of n, satisfy the conditions formulated in (1) and (2). Finally, by using a sequence running infinitely often through ω and exploiting the inductive nature of the various conditions under consideration, we can find such a B satisfying (1) and (2) for all $n < \omega$. Thus, $\mathscr{C}$ does indeed satisfy condition (iii), and the proof of Theorem 5.1 is complete.

Since $*$-purity reduces to ordinary purity in homogeneous groups, we have the following immediate consequence of our theorem.

COROLLARY 5.4 [S]. *If the homogeneous torsion-free group G is the fσ-union of a countable sequence of completely decomposable pure subgroups, then G is completely decomposable.*

Let m be an infinite cardinal. Recall that a torsion-free group G is said to be *m-separable* if every subset of cardinality less than m is contained in a completely decomposable direct summand of G.

COROLLARY 5.5. *If* $\operatorname{cof} m = \omega$, *then any m-separable group of cardinality m is completely decomposable.*

That $*$-purity in the hypotheses of Theorem 5.1 cannot be weakened to ordinary purity is confirmed by the following example.

EXAMPLE 5.6 [S]. Consider the direct sum $A = \bigoplus_{i < \omega} A_i$ where $A_0 = \langle a_0 \rangle$ and, for $i \geq 1$, $A_i = \langle a_i \rangle_*$ with $|a_i| = (\infty, 0, 0, \ldots, 0, \ldots)$. Let $p_0, p_1, p_2, \ldots, p_i, \ldots$ be the increasing sequence of primes and introduce elements $x_1, x_2, \ldots, x_i, \ldots$ such that $p_i x_i = a_0 + a_i$ for each $i \geq 1$. Then the group $G = \langle A, x_1, x_2, \ldots, x_i, \ldots \rangle$ is the ascending union of the subgroups $G_n = \langle \bigoplus_{i \leq n} A_i, x_1, x_2, \ldots, x_n \rangle$. A routine computations shows that the G_n's are pure in G. Moreover, each G_n is completely decomposable since

$$G_n = \langle b_n \rangle \oplus A_1 \oplus \cdots \oplus A_n$$

where $b_n = \lambda_1^{(n)} x_1 + \lambda_2^{(n)} x_2 + \cdots + \lambda_n^{(n)} x_n$ and the integers $\lambda_i^{(n)}$ are chosen so that $1 = \sum_{i=1}^{n} \lambda_i^{(n)} (p_1 p_2 \cdots p_n / p_i)$. Let τ denote the type determined by $(\infty, 0, 0, \ldots, 0, \ldots)$ and observe because of the purity and form of the G_n's that $G(\tau) = \bigoplus_{i \geq 1} A_i$. Now if G were to be completely decomposable, then we would have a direct decomposition $G = B \oplus C(\tau)$ where B is necessarily a rank one group. But since G is the ascending union of the pure subgroups G_n, the form of the G_n's forces us to the conclusion that B must be cyclic. This, however, is absurd since the nonzero element $a_0 + G(\tau)$ is divisible in $G/G(\tau)$ by each of the primes $p_1, p_2, \ldots, p_i, \ldots$.

6. A deficiency of $*$-purity. In this section, we show that our main result cannot be carried over to $*$-purity. In other words, a $*$-pure subgroup of a k-group need not be a k-group, nor must a $*$-pure subgroup of a separable group again be separable. In fact, we will construct an example of a $*$-pure subgroup of a countable completely decomposable group that is not itself completely decomposable. Indeed, the subgroup we construct will have no nonzero primitive elements at all! This demonstrates that $*$-purity is much weaker than Σ-purity.

The following proposition is well known; at least, it is a consequence of elementary set theory.

PROPOSITION 6.1. *Let $\mathscr{F}$ denote the (countable) collection of all finite subsets of the set P of primes. Let $\{F_n\}_{n < \omega}$ be a sequence such that (1) $F_n \in \mathscr{F}$, and (2) for each $F \in \mathscr{F}$, there are infinitely many n for which $F_n = F$. There exists a sequence $\{E_n\}_{n < \omega}$ that satisfies the following conditions whenever $i, j < \omega$.*
  (1) $E_i \subseteq P$.
  (2) E_i *is infinite and* $E_i \supseteq F_i$.
  (3) $E_i \cap E_j$ *is finite when* $i \neq j$.
  (4) $E_k \cap E_l$ *properly contains* $E_i \cap E_j$ *when* $i \neq j$ *and* $k > \max\{i, j\}$.

We are now ready to construct the example referred to above and prove the following.

THEOREM 6.2. *A countable ∗-pure subgroup of a completely decomposable group need not be completely decomposable. Indeed, such a subgroup need not possess primitive elements.*

PROOF. Select the sequence $\{E_n\}_{n < \omega}$ so that it satisfies the conditions (1)–(4) of the preceding proposition. Let s_n be the idempotent height sequence (consisting of 0's and ∞'s) such that s_n assumes the value ∞ at p if and only if $p \in E_n$. Choose a rank one group $G_n \subseteq Q$ so that the height sequence of g_n in G_n is precisely s_n. Let $G = \bigoplus_{n < \omega} G_n$. Let ϕ_n be the map from G_n to Q that maps g_n onto 1; and let ϕ: $G \to Q$ be the induced map, $\phi = \Sigma \phi_n$. Finally, denote the kernel of ϕ by H.

Observe that $H = \ker \phi$ is pure in G since ϕ maps G into the torsion-free group Q. We claim that H is ∗-pure in G. In order to show H is ∗-pure in G, we need to verify
  (i) $H \cap G(s) = H(s)$,
  (ii) $H \cap G(s^*) = H(s^*)$, and
  (iii) $H \cap G(s^*, p) = H(s^*, p)$.
We know already that condition (i) is satisfied since H is pure in G. We intend to verify that if $x \in H$ and $x \neq 0$, then the following holds.
  (iv) $x \in H(|x|^*)$.
This will not only yield immediately conditions (ii) and (iii), but in addition it will preclude x from being primitive.

Suppose $x \in H$ and $x \neq 0$. For some positive integer m, we can write (by virtue of $x \in G$)

$$mx = m_1 g_1 + m_2 g_2 + \cdots + m_r g_r,$$

where $m_i \in Z$. Recall that $g_i \in G_i$ has height exactly s_i, where s_i is idempotent and $s_i(p) = \infty$ if and only if $p \in E_i$. There is no loss of generality in assuming that $m = 1$; for if $p \mid m$ and $p \nmid m_i$, then $s_i(p) = \infty$. Therefore, $pg_i' = g_i$ where g_i' in G_i has exactly the same height sequence as g_i. Thus by replacing g_i by g_i', if necessary, we may assume that $p \mid m_i$ for each i in which case p can be cancelled from both m and the m_i's. Consequently, we assume that $x = m_1 g_1 + m_2 g_2 + \cdots + m_r g_r$. Since x, in fact, belongs to H, we know that $\Sigma_{i < r} m_i = \phi(x) = 0$. In particular, we know that at least two m_i's are different from zero since $x \neq 0$.

Observe that if we let

$$s = |m_1 g_1| \wedge |m_2 g_2| \wedge \cdots \wedge |m_r g_r|,$$

then $|x| = s$. Consequently, $|x|$ has only a finite number of ∞'s. More to the point, by invoking condition (4) of Proposition 6.1, we conclude that there exists $k > r$ such that $E_k \cap E_i$ properly contains $E_j \cap E_i$ whenever $i, j \leqslant r$ and $i \neq j$. Moreover,

$$x = m_1(g_2 - g_k) + m_2(g_2 - g_k) + \cdots + m_r(g_r - g_k)$$

because $\sum_{i \leqslant r} m_i = 0$. But $|m_i(g_i - g_k)| \geqslant s$ and $|m_i(g_i - g_k)|$ has at least one more ∞ than s (whether computed in H or G since H is pure). This implies that $x \in H(s^*)$. Since $s = |x|$, we have verified condition (iv), and the theorem is proved.

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